Numerical Optimization
Introduction

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Introduction

- People optimise.
- Investors optimise to create portfolios that avoid excessive risk and return high earnings.
- Manufacturers try to maximise production.
- Engineers modify parameters of a system to obtain maximum output.
- In nature physical systems tens to go to a state of low energy.
- Ray of light follows path that minimise its travels time.

What is Optimisation?

- Optimisation comes from the word "Optimise" which basically means "to make the best of".
- In basic applications, optimisation refers to the act or process of making something as good as it can be.
- Mathematically speaking the concept of optimisation boils down to either "to maximise" or "to minimise".
- Q. What do we maximise or minimise?
  A. A function, called the objective function.
- To optimise we must first identify some objective: "a quantitative measure of the performance of the system under study".

Examples of Objective Function

- Profit.
- Time taken to complete a task.
- Potential energy of a system.

The objective (function) depends on certain characteristics of the system, called variables or unknowns.

Examples (Variables or Unknowns)

To maximise the profit made out of a store one may look at aspects such as:

- number of hours the store is open;
- number of employee the store has;
- distance from the nearest densely populated area, etc.

Similarly, to minimise the time taken by an athlete to complete a marathon, factors such as,

- number of days per week of training;
- number of hours of sleep;
- calorie intake, etc.

Variables or Unknowns

More often than not these unknowns or variables are restricted or constrained.

- like no one can sleep for more than 24 hours in a day.
- the distance from the nearest densely populated area cannot be less than zero, etc.

Modelling

The process of identifying objectives, variables and constraints for a given problem is known as modelling.

- Construction of an appropriate model is very vital to the optimisation process (Sometimes the most crucial step).
  1. If, the model is too simple it may not give the right insights for the problem.
  2. If, its too complex, it might be too difficult to solve.
- Once an appropriate model has been formulated, an optimisation algorithm can be employed to find its solution.

Areas Where Optimisation is Required

Optimisation problems quite often arise in:

- Business:
  - Allocation of resources for logistics,
  - investment, etc.
- Science:
  - Estimation and fitting of models to measured data (these might be some parameters needed to be fixed).
  - design of experiments.
- Engineering:
  - Design and operation of technical systems/ e.g. bridges, cars, air-crafts, digital devices etc.

How to Optimise?

- There is no panacea or a silver bullet which can tackle all optimisation problems.

There is no universal optimisation algorithm.

- There are different and many algorithms to tackle optimisation problems;
- each of the algorithms are tailored to a particular type of optimisation problem.

Why Study Numerical Optimisation?

- The responsibility of choosing the algorithm that is appropriate for a specific application often falls on the user.
- This choice is important, as it determines if a problem is solved rapidly or slowly or is even solvable by the technique chosen.

An optimisation to choose an optimisation algorithm.

Q. How to know if the solution is optimal?

- One can go for derivative tests.
- life is not always so simple.
- But, in many cases, there are elegant mathematical expressions known as optimality conditions for checking the found values are indeed solution to the problem.

Exactly How To Optimise?

1. Finding the exact maxima or minima.
2. Solving it approximately to desirable accuracy.

Optimisation Algorithms (to Solve)

- Faster and more resource efficient.
- iterative: start with an initial guess and generate a sequence of improved estimates .
- The strategy used to obtain the iterates, gives the distinction between algorithms. They usually use:
  - the values of the objective function f;
  - the constraints C;
  - sometimes (possibly) the first and second derivatives of these functions.
- Some algorithms accumulate information gathered at previous iterations while others use only local information from the current point.

What is a good algorithm?

Regardless of all the nity-gritties a good algorithm should posses the following few characters:

1. **Robustness:**
   They should be effective on a wide variety of problems, in at-least a certain class of reasonable choice of initial values.
2. **Efficiency:**
   They should not require too much compute time or storage.
3. **Accuracy:**
   They should be able to identify solutions to good precission, without being perturbed by round-off errors due to the computer or being overly sensitive to small errors in the fed data, when implemented.

Types of Optimisation Problems

- Based on nature of the function at hand to be optimised:
  1. Continuous optimisation.
  2. Discrete optimisation.
  3. Stochastic optimisation.
- Based on restriction on the function to be optimised:
  1. Un-constrained optimisation.
  2. constrained optimisation.
- Based on the nature of the domain under consideration:
  1. Local optimisation.
  2. Global optimisation.