

# Numerical Optimization Mathematical Formulation

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# Mathematical Formulation

## Definition

Optimisation is the minimisation or maximisation of a function subject to constraints on its variables.

## The Mathematical Framework (Components of an Optimisation Problem)

- $X$  is a vector of variables, also called unknowns or parameters;
- $f$  is the objective function; a function of  $X$  that we want to optimise (maximise or minimise).
- $C$  is the vector function of constraints that must be satisfied by the unknowns  $X$ .

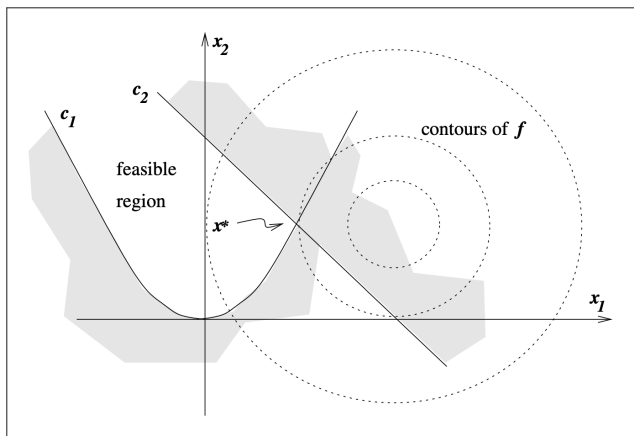
# Formal Definition

The optimisation problem can be stated as:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad \begin{cases} c_i(x) = 0, & i \in \mathcal{E} \\ c_j(x) \geq 0, & j \in \mathcal{I} \end{cases} \quad (1)$$

- $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , (a scalar valued function)
- $c_j : \mathbb{R}^n \rightarrow \mathbb{R}$
- $\mathcal{E}$ : set of indices for equality constraints
- $\mathcal{I}$ : set of indices for inequality constraints

# Optimisation Problem



**Figure:** (a) Shows the contours of objective function  $f$  and (b) feasible region i.e the region satisfying all the constraints.

# A Simple Example

$$\min_{x \in \mathbb{R}^2} (x_1 - 2)^2 + (x_2 - 1)^2 \quad \text{subject to} \quad \begin{cases} x_1^2 - x_2 \leq 0, \\ x_1 + x_2 \leq 2. \end{cases} \quad (2)$$

The above problem can be morphed into the following:

$$f(x) = (x_1 - 2)^2 + (x_2 - 1)^2, \quad \textit{the objective function}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \textit{unknowns or variables}$$

$$c(x) = \begin{bmatrix} c_1(x) \\ c_2(x) \end{bmatrix} = \begin{bmatrix} -x_1^2 + x_2 \\ -x_1 - x_2 + 2 \end{bmatrix}, \quad \textit{inequality constraints}$$

$$\mathcal{I} = \{1, 2\}, \quad \mathcal{E} = \phi$$

# Modelling an Optimisation Problem

## A transport problem:

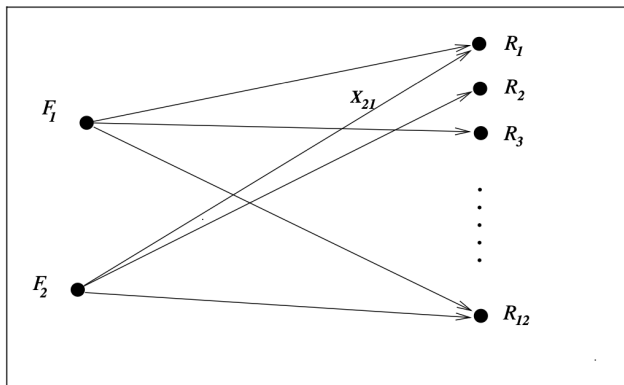
A chemical company has two factories  $F_1$  and  $F_2$  and a dozen of retail outlets  $R_1, \dots, R_{12}$ .

- $F_i$  produces  $a_i$  tons of chemicals per week i.e.  $a_i$  is the capacity of the plant.
- Each outlet  $R_j$  has a weekly demand of  $b_j$  tons of chemicals.
- cost of shipping one ton from  $F_i$  to  $R_j$  is  $c_{ij}$ .

## Challenge:

Figure out how much product to ship from each factory to each outlet while satisfying all requirements but still minimising the costs.

# TRANSPORTATION PROBLEM



**Figure:** A transportation problem: Showing the no. of tons shipped from factory  $F_i$  to outlet  $R_j$

# Modelling the Transport Problem

## Objective:

- Let  $x_{ij}$  be variables: no. of tons of chemicals shipped from  $F_i$  to outlet  $R_j$  ( $i = 1, 2$  and  $j = 1, 2, \dots, 12$ ).
- The objective function to be minimised can now be written as

$$f(x) = \sum c_{ij}x_{ij}$$

- Constraints:**

$$\sum_{j=1}^{12} x_{ij} \leq a_i \quad \text{for } i = 1, 2.$$

$$\sum_{i=1}^2 x_{ij} \geq b_j \quad \text{for } j = 1, \dots, 12$$

$$x_{ij} \geq 0, \text{ for } i = 1, 2 \quad j = 1, \dots, 12.$$
(3)



# Modelling the transport problem

- A more applicable model factors cost of manufacturing, profit from different stores, etc.
- Linear programming problems (LPP) (Objective and constraints linear)
- The variables in this case to be optimised were volumes to chemicals, which can take continuously positive values.
- If for example one has to figure out the manufacturing and transport of tractors from a factory to outlet, the answer 5.6 tractors to be sent to some outlet looks absurd.
- One could naively suggest to round up to the nearest integer value.
- Now, there is no guaranty that the modified solution is optimal in any sense.
- Problems of the first type are called continuous optimisation problems where as the second one falls in the category of discrete optimisation problems.

# Discrete Vs Continuous Optimisation Problems

## Continuous

$x_{ij} \in D$ ,  $D \subset \mathbb{R}$  union of connected sets.

## Discrete

$x_{ij} \in Z$ , for all  $i, j$  (integer programming problem).

or

- The restriction of the domain (atleast for the discrete case) can be added into the constraints.
- The generic term discrete optimisation stands for problems where the solution is seeked within a finite set (possibly too large though).
- Continuous problems are easier solver (owing to smoothness of function).
- one may get the idea to explore through all the values to find the optimal solution for a discrete problem, but this might not be feasible owing to the size of the domain.

*Some models may contain variables that are allowed to vary continuously and other that can only attain integer values;*

*these are referred to as mixed integer programming problems*

# Other Criterias to Group Optimisation Problems

There are many other categories into which optimisation problems can be put into, some of them are:

- 1 based on the nature of the objective function and constraints  
(*linear, non-linear, convex*)
- 2 based on number of variables (*large or small*).
- 3 the smoothness of the function (*differentiable or non-differentiable*) and so on.

*Categories are formed as well, based on (types of) algorithms that can be employed some method may be very well suited for large scale problems but not for small.*

*Some algorithm may need to use certain order derivative of the objective function (smoothness)*

# Global Vs Local Optimisation

*Local Optimisation problem: a maximum or minimum is sought in a vicinity of a particular point of interest.*

## Advantage:

Can be solved very fast, owing to its local nature.

## Disadvantage:

Suffers miserably to give any kind of indication about the global solution.

- For some application problems global solutions are highly desirable or necessary, but they are **extermely difficult to identify**.
- Convex programming (is a special case), where all local solutions are also global solutions.
- We focus on local optimisation and will just see through some global algorithms as well.

# Stochastic Vs Deterministic Optimisation

- For some optimisation problems the model could not be specified completely at the time of formulation owing to the unavailability of full picture about a few of the variables in time.
- For example in the transportation problem, let the customer demand  $b_j$  at the retail outlets cannot be specified precisely in advance.
- In practice, problems arising from **financial** and **economic models**, which depend on future interest rates and behaviour of the economy cannot be modelled simply.
- Modelers try to predict these unknown parameters with some degree of confidence by assigning some probability, based on previous experience.
- Stochastic Optimisation algorithms use these quantifications of the uncertainty to produce solutions that optimise the expected performance of the model.
- Deterministic Optimisation problems are those in which the model is fully specified.

# Convexity

- The concept of **convexity** is quite fundamental in optimisation.
- Many practical problems possess it, making them theoretically and in practice to be solved conveniently.
- The term "Convexity" can be realised for both **sets(domains)** and **functions (Objective, Constraints)**.

## Definition (Convex Set)

A set  $S \in \mathbb{R}^n$  is said to be convex, if the line segment joining any two points in  $S$  lie entirely inside  $S$  i.e.

$$x, y \in S \implies \alpha y + (1 - \alpha)x \in S \quad \text{for all } \alpha \in [0, 1] \quad (4)$$

## Example:

- 1 unit ball
- 2 Any polyhedron, which is a set defined by linear equalities and inequalities



# Convexity

## Definition (Convex function)

A function  $f : \mathbb{S} \rightarrow \mathbb{R}^n$ ,  $\mathbb{S} \subset \mathbb{R}^n$  is said to be convex, if  $\mathbb{S}$  is convex and for any two points  $x$  and  $y$  in  $\mathbb{S}$  the graph of  $f$  lies below the straight line connecting  $(x, f(x))$  and  $(y, f(y))$  in the space  $\mathbb{R}^{n+1}$ . That is we have the following:

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y), \text{ for all } \alpha \in [0, 1]$$

- When  $f$  is smooth as well as convex and the dimension  $n$  is 1 or 2, the graph of  $f$  is bowl-shaped and its contours define convex sets.
- A function  $f$  is said to be concave if  $-f$  is convex.

# A Convex Function

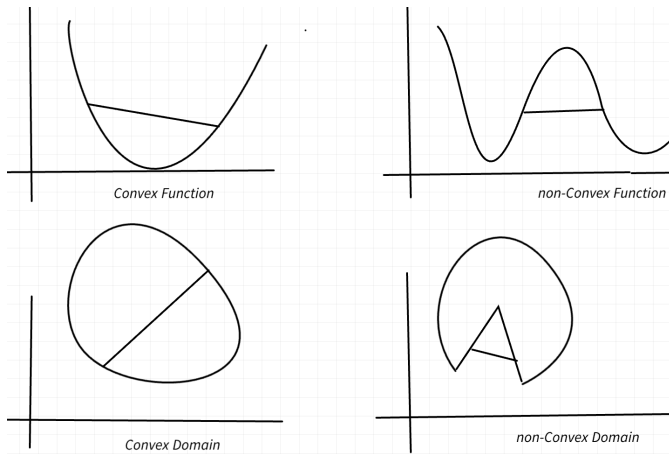
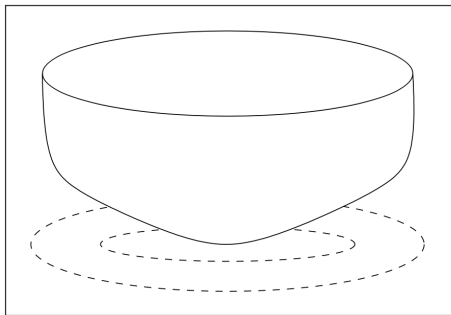


Figure: A Convex Function.

# Convex Optimisation



The convex  
function  $f(x) =$   
 $(x_1 - 6)^2 + \frac{1}{25}(x_2 - 4.5)^4$ .

Figure: A Convex Function.

- *Convex optimisation problem*: the objective function and the feasible set are both convex.
- Ex.: linear programming (LP).
- Easier to solve because every local min. is a global min.