Department of Mathematics, IIT Madras

MA-5105-Scientific Computing & Advanced Programming Lab

Practice Problems 7

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Practice Exercises: Central Limit Theorem

1. **Estimating the Probability of a Biased Coin**

   A coin lands heads with probability $p = 0.2$. We toss the coin repeatedly and estimate the probability of heads using the observed proportion.

   Let

   $$X_i = \begin{cases} 1, & \text{if the } i\text{-th toss is heads,} \\ 0, & \text{otherwise.} \end{cases}$$

   For n tosses, define the estimator

   $$\widehat{p}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

   Tasks

   (a) Using Python, generate n independent coin tosses and calculate $\widehat{p}_n$ for $n = 10, 50, 100, 500$.

   (b) Repeat the experiment several times and record the estimated proportions.

   (c) Plot a histogram of the estimates for each sample size.

   (d) Compare the histograms with a normal distribution having mean 0.2 and variance $0.16/n$.

   Explore: How does the distribution of the estimates change as the sample size increases? For which sample sizes does the normal approximation appear reasonable?

2. **Sample Means of an Exponential Distribution**

   Suppose independent observations are drawn from an exponential distribution with probability density

   $$f(x) = e^{-x}, \qquad x \geq 0.$$

   The population mean and variance are both equal to 1.

For a sample of size n, define

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Tasks

(a) Generate samples of sizes $n = 5, 20, 100$ from the exponential distribution.

(b) Calculate the sample mean for each generated sample.

(c) Repeat the experiment and collect the sample means.

(d) Plot a histogram of the sample means for each sample size.

(e) Compare the histograms with the normal distribution

$$N\left(1, \frac{1}{n}\right).$$

Explore: The original exponential distribution is right-skewed. How does the distribution of the sample means change as the sample size increases? How well does the normal approximation describe this change?

3. **How Reliable Is an Estimate?**

Suppose observations take values 0 and 1, each with probability $1/2$. We want to estimate the probability of observing a 1.

For n observations, consider the sample proportion

$$\widehat{p}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

The true probability is $p = 0.5$.

Tasks

(a) Consider sample sizes $n = 10, 50, 100, 500$.

(b) For each sample size, repeatedly generate observations and calculate the sample proportion.

(c) Calculate the fraction of estimates satisfying

$$|\widehat{p}_n - 0.5| < 0.05.$$

(d) Plot this fraction against the sample size.

(e) Compare your observations with the normal approximation

$$\widehat{p}_n \approx N\left(0.5, \frac{0.25}{n}\right).$$

Explore: How does the sample size affect the likelihood of obtaining an estimate close to the true probability? What happens if you require the estimation error to be less than 0.02 instead of 0.05?