Surface Area and Surface Integration

Saurav Samantaray

Department of Mathematics

Indian Institute of Technology Madras

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Surfaces in Space

Explicit form: $z = f(x, y)$

Implicit form: $F(x, y, z) = 0$

$->$ There is also a parametric form for surfaces that gives the position of a point on the surface as a vector function of two variables.

Parametric Form of Surfaces in Space

Suppose

$$r(u, v) = x(u, v)\; i + y(u, v)\; j + z(u, v)\; k$$

is a continuous vector function that is defined on a region R in the uv-plane and one-to-one on the interior of R

The aim is to find a double integral for calculating the area of a curved surface S based on the parametrization:

$$\begin{aligned} r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k,\ (u,v) \in D, \\ D := \{(u,v) : a \leq u \leq b, c \leq v \leq d\}. \end{aligned} \tag{1}$$

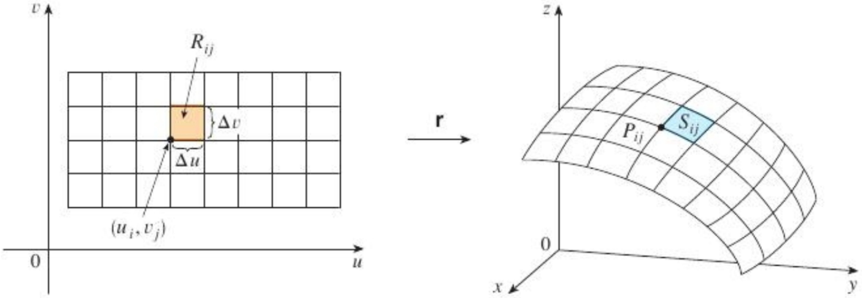
- We need S to be smooth for the construction we are about to carry out.
- Consider the following two partial derivatives:

$$\begin{aligned} r_u = \frac{\partial r}{\partial u} = \frac{\partial x}{\partial u}i + \frac{\partial y}{\partial u}j + \frac{\partial z}{\partial u}k \\ r_v = \frac{\partial r}{\partial v} = \frac{\partial x}{\partial v}i + \frac{\partial y}{\partial v}j + \frac{\partial z}{\partial v}k \end{aligned} \tag{2}$$

Smooth Surface (Definition)

A parametrized surface $r(u,v)$ (1) is smooth if r_u and r_v are continuous and $r_u \times r_v$ is never zero on the interior of the parameter domain.

- Let S be a smooth surface given parametically by (1) over a rectangular region D. (D is considered to be a rectangle for ease of derivation)
- Suppose S is covered exactly once as (u, v) vary over D.
- Divide D into small rectangle R_{ij} with the lower left corner point as $Q_{ij} = (u_i, v_j)$.
- For simplicity, let the partition be uniform with $u-$lengths as Δu and $v-$lengths as Δv.



Surface Area Derivation

- The part S_{ij} of S that corresponds to R_{ij} has the corner P_{ij} with position vector $r(u_i, v_j)$.
- The tangent vectors to S at $P_{ij} = r(u_i, v_j)$ are given by:

$$\begin{aligned} r_u^* &= r_u(u_i, v_j) = \frac{\partial x}{\partial u}(u_i, v_j)\, i + \frac{\partial y}{\partial u}(u_i, v_j)\, j + \frac{\partial z}{\partial u}(u_i, v_j)\, k \\ r_v^* &= r_v(u_i, v_j) = \frac{\partial x}{\partial v}(u_i, v_j)\, i + \frac{\partial y}{\partial v}(u_i, v_j)\, j + \frac{\partial z}{\partial v}(u_i, v_j)\, k \end{aligned} \tag{3}$$

- The tangent plane to S is the plane that contains the two tangent vectors r_u^* and r_v^*.
- The normal to S at P_{ij} is the vector $r_u^* \times r_v^*$
- Notice that since S is assumed to be smooth, the normal vector is non-zero.

Surface Area Derivation

- The part S_{ij} is a curved parallelogram on S whose sides can be approximated by the vectors

$$r_u^* \Delta u = P_{i+1,j} - P_{i,j}$$
$$r_v^* \Delta u = P_{i,j+1} - P_{i,j}$$

- Therefore, the area of S_{ij} can be approximated by:

$$\text{Area of } S_{ij} \approx |r_u^* \times r_v^*| \Delta u \Delta v$$

- Then an approximation to the area of S is obtained by summing over both indices i and j:

$$\text{Area of } S \approx \sum_j \sum_i |r_u^*(u_i, v_j) \times r_v^*(u_i, v_j)| \Delta u \Delta v \tag{4}$$

- We thus define the surface area by taking the limit of the approximated quantity in (4).

Surface Area

Let S be a smooth surface given parametrically by

$$r = x(u,v)\, i + y(u,v)\, j + z(u,v)\, k,$$

where $(u,v) \in D$, a region in the uv-plane. Suppose that S is covered exactly once as (u,v) varies over D. Then the surface area of S is given by:

$$\text{Area of } S = \int\int_D |r_u \times r_v| dA \tag{5}$$

where $r_u = x_u i + y_u j + z_u k$ and $r_v = x_v i + y_v j + z_v k$.

- Let the surface S is given by the graph of a function such as $z = f(x, y)$, where $(x, y) \in D$,
- then we take the parameters as

$$u = x,\ v = y,\ z = z(u, v) = f(x, y)$$

- That is, S is given by:
.

$$r = ui + vj + zk.$$

- Now, we have

$$r_u = i + z_u k = i + f_x k, \quad r_v = j + z_v k = i + f_y k$$

- we have

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x i - f_y j + k$$

-

$$\text{Area of } S = \int\int_D |r_u \times r_v| dA = \int\int_D (\sqrt{f_x^2 + f_y^2 + 1}) dA$$

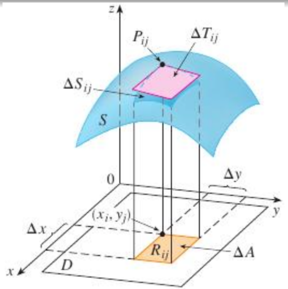
- The surface area formula can also be derived from the first principle as we had done for the parametric form.
- For this, suppose that S is given by the equation

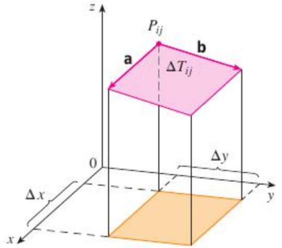
$$z = f(x, y) \text{ for } (x, y) \in D$$

- Divide D into smaller rectangles R_{ij} with area:

$$\Delta R_{ij} = \Delta x \Delta y$$

- For the corner (x_i, y_j) in R_{ij}, closest to the origin, let P_{ij} be the point $(x_i, y_j, f(x_i, y_j))$ on the surface.
- The tangent plane to S at P_{ij} is an approximation to S near P_{ij}.





- The area T_{ij} of the portion of the tangent plane that lies above R_{ij} approximates the area of S_{ij}, the portion of S that is directly above R_{ij} .
- Therefore, we define an approximation to the area of the surface S as

$$\Delta(S) = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} T_{ij}$$

- Now to compute T_{ij}, let's consider a and b to be the vectors that start at P_{ij} and lie along the sides of the parallelogram.
- Then $T_{ij} = |a \times b|$
- a can be approximated by the vector $P_{i+1,j} - P_{i,j}$, giving us:

$$a = \Delta x i + f_x(x_i, y_j)\Delta x k$$

- Similarly, b is given by:

$$b = \Delta y j + f_y(x_i, y_j)\Delta y k$$

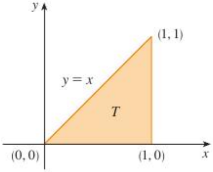
- Therefore T_{ij} an be computed as:

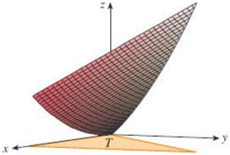
$$\begin{aligned} T_{ij} &= |a \times b| = |-f_x(x_i, y_j)i - f_y(x_i, y_j)j + k|\Delta(R_{ij}) \\ &= \sqrt{f_x^2 + f_y^2 + 1}\Delta(R_{ij}) \end{aligned}$$

- Summing over these T_{ij} and taking the limit, we obtain:

$$\text{Area of } S = \int\int_D (\sqrt{f_x^2 + f_y^2 + 1})dA \tag{6}$$

Find the surface area of the part of the surface $z = x^2 + 2y$ that lies above the triangular region in the xy-plane with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

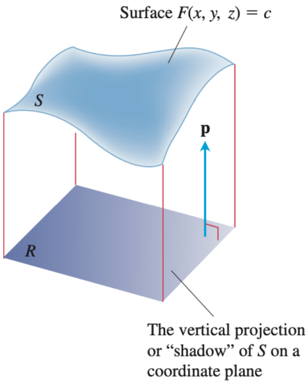




- $T = \{(x,y) : 0 \leq x \leq 1,\ 0 \leq y \leq x\}$, and $f(x,y) = x^2 + 2y$.
- The surface area is:

$$\int\int_T \sqrt{(2x)^2 + 2^2 + 1}\,dA = \int_0^1 \int_0^x \sqrt{4x^2+5}\,dy\,dx = \frac{1}{12}(27 - 5\sqrt{5}).$$

- Surfaces are often presented as level sets of a function, described by an equation such as:
$$F(x, y, z) = c, \tag{7}$$
for some constant c.
- Such a level surface does not come with an explicit parametrization, and is called an implicitly defined surface.



- Let S lies above a "shadow" region R in the plane beneath it,
- and p is a unit vector normal to the plane region R
- We assume that the surface is smooth (F is dif- ferentiable and ∇F is nonzero and continuous on S) and that $\nabla F \cdot p \neq 0$.
- so the surface never folds back over itself.

- The Implicit Function Theorem implies that S is then the graph of a differentiable function $z = f(x, y)$,
- although the function $f(x, y)$ is not explicitly known.
- Define the parameters u and v by $u = x$ and $v = y$.
- Then $z = f(u, v)$
- As was done before

$$r(u, v) = ui + vj + f(u, v)k$$

  gives a parametrization of the surface S.
- Then we have

$$r_u = i + f_x k \quad \text{and} \quad r_v = j + f_y k$$

- Now consider the partial derivatives of $F(x, y, z) = c$ to get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

- We obtain the partial derivatives as

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

- Assume that the normal vector p is the unit vector k, so the region R lies in the xy-plane.
- By assumption, we then have $\nabla F \cdot p = \nabla F \cdot k = F_z \neq 0$ on S.
- Substitution of these derivatives into the derivatives of r gives

$$r_u = i - \frac{F_x}{F_z}k \quad \text{and} \quad r_v = j - \frac{F_y}{F_z}k$$

- Therefore we have

$$\begin{aligned}
r_u \times r_v &= \frac{F_x}{F_z}i + \frac{F_y}{F_z}j + k && (F_z \neq 0) \\
&= \frac{1}{F_z}(F_x i + F_y j + F_z k) \\
&= \frac{\nabla F}{F_z} = \frac{\nabla F}{\nabla F \cdot k} \\
&= \frac{\nabla F}{\nabla F \cdot p} && (p = k)
\end{aligned}$$

Surface Area of $F(x, y, z) = c$

Let the surface S be given by $F(x, y, z) = c$. Let R be a closed bounded region which is obtained by projecting the surface to a plane whose unit normal is p. Suppose that ∇F is continuous on R and $\nabla F \cdot p \neq 0$. Then

$$\text{surface area of } S = \int\int_R \frac{|\nabla F|}{|\nabla F \cdot p|} dA \tag{8}$$

whenever possible, we project onto the coordinate planes.

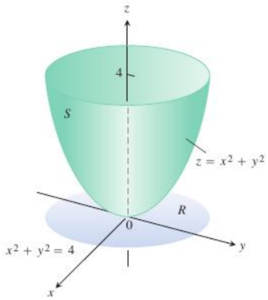
- if $F(x, y, z) = c$ could be written as $f(x, y) = z$ as a result of the implicit function theorem we have

$$\frac{|\nabla F|}{|\nabla F \cdot p|} = \frac{\sqrt{f_x^2 + f_y^2 + 1}}{1^2}$$

which is the integrand in the surface area formula.

Example

Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 = z$ by the plane $z = 4$



- Surface S is given by $f(x, y, z) = x^2 + y^2 - z = 0$.
- Projection to xy-plane to get the region R as $x^2 + y^2 \leq 4$.
- $\nabla f = 2xi + 2yj - k$
- $|\nabla f| = \sqrt{1 + 4x^2 + 4y^2}$
- Here $p = k, \implies |\nabla f \cdot p| = 1$

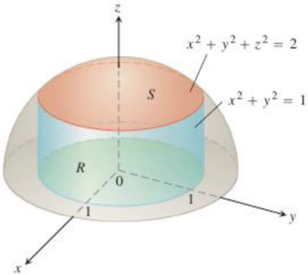
- R can be expressed in polar co-ordinate given by $x = r\cos\theta$, $y = r\sin\theta$, $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$.
- So the surface area of S is

$$\int\int_R \sqrt{1 + 4x^2 + 4y^2}dA = \int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \frac{\pi}{6}(17\sqrt{17} - 1).$$

Example

Find the surface area of the cap cut by the cylinder $x^2 + y^2 = 1$ from the hemisphere

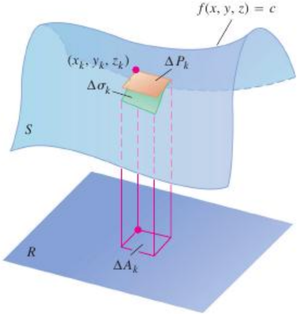
$x^2 + y^2 + z^2 = 2$, $z \geq 0$.



- The surface projected on xy-plane gives R as the disk $x^2 + y^2 \leq 1$.
- The surface is $f(x, y, z) = 2$, where $f(x, y, z) = x^2 + y^2 + z^2$.
- $\nabla f = 2xi + 2yj + 2zk$
- $|\nabla f| = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{2}$.
- $p = k \implies |\nabla f \cdot p| = |2z| = 2z$

- To compute the flow of a liquid across a curved membrane, or the total electrical charge on a surface, we need to integrate a function over a curved surface in space.
- Such a *surface integral* is the two-dimensional extension of the *line integral* concept used to integrate over a one-dimensional curve.
- Like line integrals, surface integrals arise in two forms.
    1. When we integrate a scalar function over a surface, such as integrating a mass density function defined on a surface to find its total mass.
    2. surface integrals of vector fields
- The first form corresponds to line integrals of scalar functions.
- The second form is analogous to the line integrals for vector fields.
- An example of this form occurs when we want to measure the net flow of a fluid across a surface submerged in the fluid.

- let $g(x, y, z)$ ($g : D \subset \mathbb{R}^3 \to \mathbb{R}$) is a function is defined over a surface S.
- Let the surface S be given by $f(x, y, z) = c$.
- To compute the integral of g, over S we need to look at area elements on S.
- In turn we look at the region in the plane over which the surface S is considered.
- Divide the region R into smaller rectangles of area ΔA_k
- Consider the area of the corresponding portion of the surface as $\Delta\sigma_k$.



- Then we have

$$\Delta\sigma_k = \left(\frac{|\nabla f|}{|\nabla f \cdot p|}\right)_k \Delta A_k$$

- Assuming that g is nearly constant on the smaller surface fragment σ_k, we form the sum

$$\sum_k g(x_k, y_k, z_k)\Delta\sigma_k \approx \sum_k g(x_k, y_k, z_k)\left(\frac{|\nabla f|}{|\nabla f \cdot p|}\right)_k \Delta A_k$$

- If the above sum converges, then we define that limit as the integral of g over the surface S

Definition

- Let Sbe a surface given by $f(x, y, z) = c$.
- Let the projection of S onto a plane with unit normal p be the region R.
- Let $g(x, y, z)$ be a scalar valued function defined over S.

Then the surface integral of g over S is:

$$\int\int_s g\, d\sigma = \int\int_R g(x, y, z)\frac{|\nabla f|}{|\nabla f \cdot p|} dA. \tag{9}$$

We say $d\sigma = \frac{|\nabla f|}{|\nabla f \cdot p|} dA$

Linearity

- If the surface S can be represented as a union of non-overlapping smooth surfaces $S_1, \ldots, S_n$, then

$$\int\int_s g\, d\sigma = \int\int_{s_1} g\, d\sigma + \ldots + \int\int_{s_n} g\, d\sigma.$$

- If $g(x, y, z) = g_1(x, y, z) + \ldots + g_m(x, y, z)$ over the surface S, then

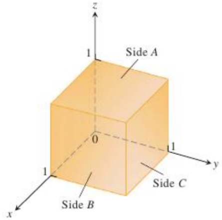
$$\int\int_s g\, d\sigma = \int\int_s g_1\, d\sigma + \ldots + \int\int_s g_m\, d\sigma.$$

- Similarly, if $g(x, y, z) = kh(x, y, z)$ holds for a constant k, over S, then

$$\int\int_s g\, d\sigma = k \int\int_s h\, d\sigma$$

Example

Integrate $g(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$,and $z = 1$.



- We integrate g over the six surfaces and add the results.
- As $g = xyz$ is zero on the co-ordinate planes, we need integrals on sides A, B and C.
- Side A is the surface $z = 1$ defined on the region $R_A : 0 \leq x \leq 1, 0 \leq y \leq 1$ on the xy-plane.
- For this surface and the region,

$$p = k, \ \nabla f = k, \ |\nabla f \cdot p| = 1$$

- Since $z = 1$ we have $g(x, y, z)| = xyz|_{z=1} = xy$.
- the surface integral is given by

$$\int\int_A g(x, y, z) \, d\sigma = \int\int_{R_A} xy \frac{|\nabla f|}{|\nabla f \cdot p|} dA = \int_0^1 \int_0^1 xydxdy = \int_0^1 \frac{y}{2} = \frac{1}{4}$$

- Similarly, compute $\int\int_B g(x, y, z) \, d\sigma$ and $\int\int_C g(x, y, z) \, d\sigma$ to finally write

$$\int\int g(x, y, z) \, d\sigma = \int\int g(x, y, z) \, d\sigma + \int\int g(x, y, z) \, d\sigma + \int\int g(x, y, z) \, d\sigma$$

Example

Evaluate the surface integral of $g(x, y, z) = x^2$ over the unit sphere.

- The surface can be divided into the upper hemisphere and the lower hemisphere.
- Let S be the upper hemisphere $f(x, y, z) := x^2 + y^2 + z^2 = 1$, $z \geq 0$.
- Its projection on the xy-plane is the region

$$R : x = r\cos\theta, y = r\sin\theta, 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$$

- Here, $p = k$, $\nabla f = 2\sqrt{x^2 + y^2 + z^2} = 2$.
- $|\nabla f \cdot p| = 2|z| = 2\sqrt{1 - (x^2 + y^2)} = 2\sqrt{1 - r^2}$.
- We, have

$$\int\int_S x^2 \, d\sigma = \int\int_R x^2 \frac{|\nabla f|}{|\nabla f \cdot p|} dA = \int\int_R \frac{x^2}{\sqrt{1 - r^2}} dA$$

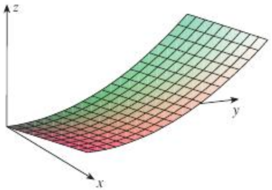
$$= \int_0^{2\pi}\int_0^1 \frac{r^2\cos^2\theta}{\sqrt{1 - r^2}} r \, dr \, d\theta = \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^1 \frac{r^3}{\sqrt{1 - r^2}} \, dr = \frac{2\pi}{3}.$$

Since the integral of x^2 on the upper hemisphere is equal to that on the lower hemisphere, the required integral is $2 \times \frac{2\pi}{3} = \frac{4\pi}{3}$.

A simplification

- When $p = k$, that is, when the region R is obtained by projecting the surface S onto the xy-plane, then we have
$$\frac{|\nabla f|}{|\nabla f \cdot p|} = \sqrt{1 + z_x^2 + z_y^2}$$
- Therefore, if the surface $f(x, y, z) = c$ can be written explicitly by $z = h(x, y)$, then the surface integral takes the form
$$\int\int_S g(x, y, z)\, d\sigma = \int\int_R g(x, y, h(x, y))\sqrt{1 + h_x^2 + h_y^2}\, dxdy,$$
where R is the region obtained by projecting S on to the xy-plane.
- Similarly, if the surface can be written as $y = h(x, z)$ and R is obtained by projecting S onto the xz-plane, then
$$\int\int_S g(x, y, z)\, d\sigma = \int\int_R g(x, h(x, z), z)\sqrt{1 + h_x^2 + h_z^2}\, dxdz.$$
- if the surface can be written as $x = h(y, z)$ and R is obtained by projecting S onto the yz-plane, then
$$\int\int_S g(x, y, z)\, d\sigma = \int\int_R g(h(y, z, y, z)\sqrt{1 + h_y^2 + h_z^2}\, dydz.$$

Evaluate $\int\int_S y \, d\sigma$, where S is the surface $z = x + y^2$, where $0 \leq x \leq 1$ and $0 \leq y \leq 2$.



- Projecting the surface to xy-plane, we obtain the region R as the rectangle $0 \leq x \leq 1$, $0 \leq y \leq 2$.
- The surface is given by $z = h(x, y) = x + y^2$.
- So the surface integral is

$$\int\int_S y \, d\sigma = \int\int_R y\sqrt{1 + 1 + (2y)^2} \, dA = \int_0^1 \int_0^2 \sqrt{2}y\sqrt{(1 + 2y^2)} \, dy \, dx$$
$$= \frac{13\sqrt{2}}{3}.$$

- Suppose the surface S is given in a parameterized form:

$$r(u,v) = x(u,v)i + y(u,v)j + z(u,v)k$$

  where (u,v) ranges over the region Din the uv-plane.
- Here, a change of variable happens. Then

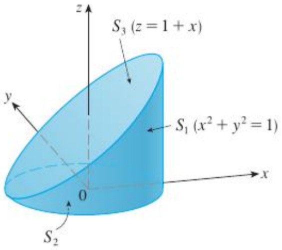
$$d\sigma = |r_u \times r_v| dudv$$

  where $r_u = x_u i + y_u j + z_u k$ and $r_v = x_v i + y_v j + z_v k$
- Therefore, we have

$$\int\int_S g(x,y,z)\, d\sigma = \int\int_D g(r(u,v))|r_u \times r_v| dudv$$

Evaluate $\int\int_S z\, d\sigma$, where S is the surface whose sides S_1, S_2, S_3 are:

- S_1 is given by the cylinder $x^2 + y^2 = 1$,
- bottom S_2 is the disk $x^2 + y^2 \leq 1, z = 0$, and,
- whose top S_3 is part of the plane $z = 1 + x$ that lies above S_2.



- S_1 is given by

$$xi + yj + zk, \text{ on the region } D, \text{ given by}$$

$$x = \cos\theta, y = \sin\theta, 0 \leq \theta \leq 2\pi \text{ and } 0 \leq z \leq 1 + x \implies 0 \leq z \leq 1 + \cos\theta$$

- Then $|r_\theta \times r_z| = |\cos\theta i + \sin\theta j| = 1$
- We get,

$$\int\int_{S_1} z\, d\sigma = \int\int_D z|r_\theta \times r_z| dA = \int_0^{2\pi}\int_0^{1+\cos\theta} z\, dz\, d\theta = \int_0^{2\pi} \frac{(1+\cos\theta)^2}{2}\, d\theta = \frac{3\pi}{2}.$$

- S_2 lies in the plane $z = 0$. Hence,

$$\int\int_{S_2} z\, d\sigma$$

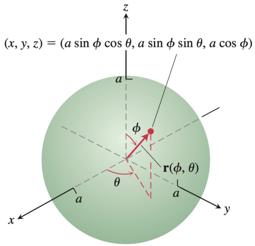
- S_3 lies above the unit disk and lies in the plane $z = 1 + x$. So,

$$\begin{aligned}\int\int_{S_3} z\, d\sigma &= \int\int_D (1+x)\sqrt{1+z_x^2+z_y^2}\,dA \\ &= \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{1+1+0}\; r\, dr\, d\theta = \sqrt{2}\pi.\end{aligned}$$

- Hence,

$$\int\int_S z\, d\sigma = \int\int_{S_1} z\, d\sigma + \int\int_{S_2} z\, d\sigma + \int\int_{S_3} z\, d\sigma = \frac{3\pi}{2} + 0 + \sqrt{2}\pi$$

- The curve C in a line integral inherits a natural orientation from its parametrization $r(t)$ because the parameter belongs to an interval $a \leq t \leq b$ directed by the real line.
- The unit tangent vector T along C points in this forward direction.
- For a surface S, the parametrization $r(u, v)$ gives a vector $r_u \times r_v$ that is normal to the surface,
- but if S has two "sides," then at each point the negative $-(r_u \times r_v)$ is also normal to the surface,
- so we need to choose which direction to use.

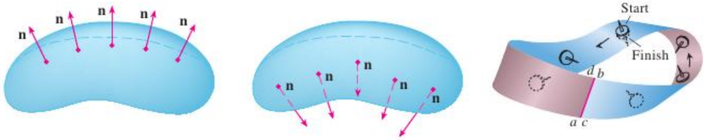


- If we look at the sphere, at any point on the sphere there is a normal vector pointing inward toward the center of the sphere and another opposite normal pointing outward.
- When we specify which of these normals we are going to use consistently across the entire surface, the surface is given an *orientation*.

Orientable Surface

Definition

A smooth surface S is called orientable (or two-sided) if it is possible to define a field of unit normal vectors $\hat{n}$ on S which varies continuously with position. Once such normal vectors are chosen, the surface is considered an oriented surface



- The Möbius band is not orientable.
- No matter where we start to construct a continuous unit normal field moving the vector continuously around the surface in the manner shown will return it to the starting point with a direction opposite to the one it had when it started out.
- The vector at that point cannot point both ways and yet it must if the field is to be continuous.

- If the surface S is given by $z = f(x, y)$, then we take its orientation by considering the unit normal vectors
$$\hat{n} = \frac{-f_x i - f_y j + k}{\sqrt{1 + f_x^2 + f_y^2}}$$
- If S is a part of a level surface $g(x, y, z) = c$, then we may take
$$\hat{n} = \frac{\nabla g}{|\nabla g|}$$
- If S is given parametrically as$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k$, then
$$\hat{n} = \frac{r_u \times r_v}{|r_u \times r_v|}$$
- Conventionally, the outward direction is taken as the positive direction.
- But, sometimes we may take negative sign if it is preferred.

Examples of Parametrization

1. The cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$, can be parametrized by $x = r\cos\theta$, $y = r\sin\theta$ and $z = r$, $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Then its vector form is:

$$r(r, \theta) = r\cos\theta i + r\sin\theta j + rk.$$

2. The sphere $x^2 + y^2 + z^2 = a^2$ can be parametrized by $x = a\cos\theta\sin\phi$, $y = a\sin\theta\sin\phi$ and $z = a\cos\phi$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. In vector form the parametrization is

$$r(\theta, \phi) = a\cos\theta\sin\phi i + a\sin\theta\sin\phi j + a\cos\phi k.$$

3. The cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq 5$ can be parametrized by

$$r(\theta, z) = a\cos\theta i + a\sin\theta j + zk.$$

Definition

Let F be a vector field in three-dimensional space with continuous components defined over a smooth surface S having a chosen field of normal unit vectors $\hat{n}$ orienting S. Then the surface integral of F over S is

$$\int\int_S F \cdot \hat{n}\, d\sigma. \tag{10}$$

It is also called the flux of F across S.

- The flux is the integral of the scalar component of F along the unit normal to the surface.
- Thus in a flow, the flux is the net rate at which the fluid is crossing the surface S in the chosen positive direction.

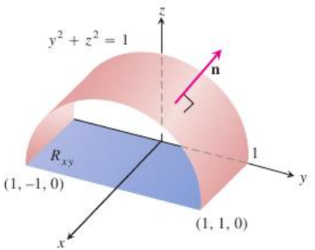
- If S is part of a level surface $g(x,y,z)=c$, which is defined over the region D, then $d\sigma = \frac{\nabla g}{|\nabla g|}$ then $d\sigma = \frac{\nabla g}{|\nabla g \cdot p|}\, dA$. So the flux across S is

$$\int\int_S F \cdot \hat{n}\, d\sigma = \int\int_S F \cdot \frac{\nabla g}{|\nabla g|}\, d\sigma = \int\int_D F \cdot \frac{\nabla g}{|\nabla g \cdot p|}\, dA$$

- If S is parametrized by $r(u,v)$, where D is the region in uv-plane, then $d\sigma = |r_u \times r_v|\, dA$. So the flux across S is

$$\int\int_S F \cdot \hat{n}\, d\sigma = \int\int_S F \cdot \frac{r_u \times r_v}{|r_u \times r_v|}\, d\sigma = \int\int_D F(r(u,v))(r_v \times r_v)\, dA.$$

Find the flux of $F = yzj + z^2k$ outward through the surface S which is cut from the cylinder $y^2 + z^2 = 1$, $z \geq 0$ by the planes $x = 0$ and $x = 1$.



- S is given by $g(x, y, z) := y^2 + z^2 - 1 = 0$, defined over the rectangle $R = R_{xy}$ as in the figure.
- The outward unit normal is $\hat{n} = +\frac{\nabla g}{|\nabla g|} = yj + zk$.
- Here $p = k$. So, $d\sigma = \frac{|\nabla g|}{|\nabla g \cdot k|} dA = \frac{1}{2z} dA$.
- Therefore, outward flux through S is

$$\int\int_S F \cdot \hat{n}\, d\sigma = \int\int_S z\, d\sigma = \int\int_R z \frac{1}{2z}\, dA = \frac{1}{2} \text{ Area of } R = 1$$

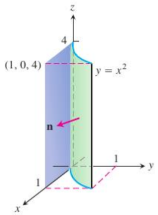
Example

Find the flux of the vector field $F = zi + yj + xk$ across the unit sphere.

- If no direction of the normal vector is given and the surface is a closed surface, we take $\hat{n}$ in the positive direction, which is directed outward.
- Using the spherical co-ordinates, the unit sphere S is parametrized by
$$r(\phi, \theta) = \sin\phi\cos\theta i + \sin\phi\sin\theta j + \cos\phi k,$$
where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$ gives the region D.
- $$F(r(\phi, \theta)) = \cos\phi i + \sin\phi\sin\theta j + \sin\phi\cos\theta k.$$
- $$r_\phi \times r_\theta = \sin^2\phi\cos\theta i + \sin^2\phi\sin\theta j + \sin\phi\cos\phi k.$$
- $$\int\int_S F\cdot\hat{n}\,d\sigma = \int\int_S F(r(u,v))\cdot(r_v \times r_v)\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}(2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\,d\theta\,d\phi$$
$$= 2\int_0^{\pi}\sin^2\phi\cos\phi\,d\phi\int_0^{2\pi}\cos\theta\,d\theta + \int_0^{\pi}\sin^3\pi\,d\phi\int_0^{2\pi}\sin^2\theta\,d\theta$$
$$= 0 + \int_0^{\pi}\sin^3\pi\,d\phi\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{4\pi}{}$$

Example

Find the surface integral of $F = yzi + xj - z^2k$ over the portion of the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$.



- We assume the positive direction of the normal $\hat{n}$.
- On the surface,we have $x = x$, $y = x^2$, $z = z$ giving the parametrization as $r(x, z) = xi + x^2j + zk$,
- and D is given by $0 \le x \le 1$, $0 \le z \le 4$.
- On the surface S F is given as $F = x^2zi + xj - zk$.

$$\begin{aligned}\int\int_S F \cdot \hat{n}\, d\sigma &= \int\int_D F \cdot (r_x \times r_z)\, dx\, dz \\ &= \int\int_D (x^2zi + xj - z^2k) \cdot (2xi - j)\, dx\, dz \\ &= \int_0^4\int_0^1 (2x^3 - x)\, dx\, dz = \int_0^4 (z-1)/2\, dz = 2.\end{aligned}$$

- If S is given by $z = f(x, y)$, then think of x, y as the parameters u and v. We have

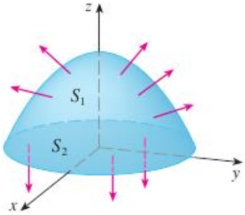
$$F(x, y) = M(x, y)i + N(x, y)j + P(x, y)k \quad \text{and,}$$

$$r = xi + yj + f(x, y)k$$

- Then, $r_x \times r_y = (i + f_x k) \times (j + f_y k) = -f_x i - f_y j + k$.
- Therefore, the flux is

$$\int\int_S F \cdot \hat{n}\, d\sigma = \int\int_D F \cdot (r_x \times r_y)\, dx\, dy = \int\int_D (-Mf_x - Nf_y + P)\, dx\, dy$$

Evaluate $\int\int_S F\cdot\hat{n}\,d\sigma$ where S $F = yi + xj + zk$ and S is the boundary of the solid enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.



- The surface S has two parts: the top portion S_1 and the base S_2.
- Since S is a closed surface, we consider its outward normal $\hat{n}$. Projections of both S_1 and S_2 on xy-plane are D, the unit disk.
- By the simplified formula for the flux, we have

$$\int\int_{S_1} F\cdot\hat{n}\,d\sigma = \int\int_D (-Mf_x - Nf_y + P)\,dx\,dy$$
$$= \int\int_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2]\,dx\,dy$$
$$= \int_0^{2\pi}\int_0^1 (1 + 4r^2\cos\theta\sin\theta - r^2)\,r\,dr\,d\theta = \int_0^{2\pi}(\frac{1}{4} + \cos\theta\sin\theta)\,d\theta = \frac{\pi}{2}$$

- The disk S_2 has positive direction, when $\hat{n} = -k$. Thus

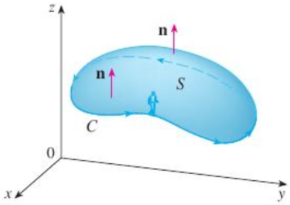
$$\int\int_{s_2} F \cdot \hat{n}\, d\sigma = \int\int_{s_2} (-F \cdot k\, d\sigma = \int\int_{D} (-z)\, dx\, dy = 0$$

  since on D for S_2, $z = 0$.

- Therefore,

$$\int\int_{s} F \cdot \hat{n}\, d\sigma = \int\int_{s_1} F \cdot \hat{n}\, d\sigma + \int\int_{s_2} F \cdot \hat{n}\, d\sigma = \frac{\pi}{2}.$$

- Consider an oriented surface with a normal vector $\hat{n}$.
- Call the boundary curve of S as C.
- The orientation of S induces a positive orientation of the boundary of S.
- If we walk in the positive direction of C keeping our head pointing towards $\hat{n}$, then S will be to your left.



- Recall that Green's theorem relates a double integral in the plane to a line integral over its boundary.
- We will have a generalization of this to 3 dimensions.
- Write the boundary curve of a given smooth surface as ∂S.
- The boundary is assumed to be a closed curve, positively oriented unless specified otherwise.

Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve ∂S with positive orientation. Let $F = Mi + Nj + Pk$ be a vector field with M, N, P having continuous partial derivatives on an open region in space that contains S. Then

$$\oint_{\partial S} F \cdot d\vec{r} = \int\int_S \mathbf{curl}$$